ON THE GREEN FUNCTIONS OF GRAVITATIONAL RADIATION THEORY

Giampiero Esposito

Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Complesso Universitario di Monte S. Angelo, Via Cintia, Edificio N', 80126 Napoli, Italy

Università di Napoli Federico II, Dipartimento di Scienze Fisiche, Complesso Universitario di Monte S. Angelo, Via Cintia, Edificio N', 80126 Napoli, Italy

Abstract. Previous work in the literature has studied gravitational radiation in black-hole collisions at the speed of light. In particular, it had been proved that the perturbative field equations may all be reduced to equations in only two independent variables, by virtue of a conformal symmetry at each order in perturbation theory. The Green function for the perturbative field equations is here analyzed by studying the corresponding second-order hyperbolic operator with variable coefficients, instead of using the reduction method from the retarded flat-space Green function in four dimensions. After reduction to canonical form of this hyperbolic operator, the integral representation of the solution in terms of the Riemann function is obtained. The Riemann function solves a characteristic initial-value problem for which analytic formulae leading to the numerical solution are derived.

1. Introduction

The construction of suitable inverses of differential operators lies still at the very heart of many profound properties in classical and quantum field theory. For example, the theory of small disturbances in local field theory can only be built if suitable invertible operators are considered [1]. In a path-integral formulation, these correspond to the gauge-field and ghost operators, respectively [2, 3]. Moreover, the Peierls bracket on the space of physical observables, which is a Poisson bracket preserving the invariance under the full infinite-dimensional symmetry group of the theory, is obtained from the advanced and retarded Green functions of the theory via the supercommutator function [1–4], and leads possibly to a deeper approach to quantization. Last, but not least, a perturbation approach to classical general relativity relies heavily on a careful construction of Green functions of operators of hyperbolic [5–8] and elliptic [9, 10] type. In particular, following [5–8], we shall be concerned with the axisymmetric collision of two black holes travelling at the speed of light, each described in the centre-of-mass frame before the collision by an impulsive plane-fronted shock wave with energy μ. One then passes to a new frame to which a large Lorentz boost is applied. There the energy $\nu = \mu e^{\alpha}$ of the incoming shock 1 obeys $\nu >> \lambda$, where $\lambda = \mu e^{-\alpha}$ is the energy of the incoming shock 2 and $\mathrm{e}^{\alpha} \equiv \sqrt{\frac{1+\beta}{1-\beta}}$ (β being the usual relativistic parameter). In the boosted frame, to the future of the strong shock 1, the metric can be expanded in the form [6, 8]

$$g_{ab} \sim \nu^2 \left[\eta_{ab} + \sum_{i=1}^{\infty} \left(\frac{\lambda}{\nu} \right)^i h_{ab}^{(i)} \right] \tag{1.1}$$

where η_{ab} is the standard notation for the Minkowski metric. The task of solving the Einstein field equations becomes then a problem in singular perturbation theory, having to find $h_{ab}^{(1)}, h_{ab}^{(2)}, \ldots$ by solving the linearized field equations at first, second, ... order respectively in $\frac{\lambda}{\nu}$, once that characteristic initial data are given just to the future of the strong shock 1. The perturbation series (1.1) is physically relevant because, on boosting back to the centre-of-mass frame, it is found to give an accurate description of space-time geometry where gravitational radiation propagates at small angles away from the forward

symmetry axis $\hat{\theta}=0$. The news function c_0, which describes gravitational radiation arriving at future null infinity in the centre-of-mass frame, is expected to have the convergent series expansion [6, 8]

$$c_0(\hat{\tau},\hat{\theta})=\sum_{n=0}^{\infty} a_{2n}(\hat{\tau}/\mu)(\sin\hat{\theta})^{2n} \tag{1.2}$$

with $\hat{\tau}$ a suitable retarded time coordinate, and μ the energy of each incoming black hole in the centre-of-mass frame. In [6, 8] a very useful analytic expression of $a_2(\hat{\tau}/\mu)$ was derived, exploiting the property that perturbative field equations may all be reduced to equations in only two independent variables, by virtue of a remarkable conformal symmetry at each order in perturbation theory. The Green function for perturbative field equations was then found by reduction from the retarded flat-space Green function in four dimensions.

However, a *direct* approach to the evaluation of Green functions appears both desirable and helpful in general, and it has been our aim to pursue such a line of investigation. For this purpose, reduction to two dimensions with the associated hyperbolic operator is studied again in section 2. Section 3 performs reduction to canonical form with the associated Riemann function. Equations for the Goursat problem obeyed by the Riemann function are derived in section 4, while the corresponding numerical algorithm is discussed in section 5.

2. Reduction to two dimensions and the associated operator

As is well known from the work in [6] and [8], the field equations for the first-order correction $h_{ab}^{(1)}$ in the expansion (1.1) are particular cases of the general system given by the flat-space wave equation (here $u \equiv \frac{1}{\sqrt{2}}(z+t), v \equiv \frac{1}{\sqrt{2}}(z-t)$)

$$\Box\psi = 2\frac{\partial^2\psi}{\partial u\partial v}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}=0 \tag{2.1}$$

supplemented by the boundary condition

$$\psi(u=0)={\rm e}^{im\phi}\rho^{-n}f[8\log(v\rho)-\sqrt{2}v] \tag{2.2a}$$

$$f(x) = 0 \ \ \forall x < 0. \tag{2.2b}$$

Moreover, ψ should be of the form $\mathrm{e}^{im\phi}\rho^{-n}\chi(q,r)$ for $u \geq 0$, where

$$q \equiv u\rho^{-2} \tag{2.3}$$

$$r \equiv 8\log(\nu\rho) - \sqrt{2}v. \tag{2.4}$$

For the homogeneous wave equation (2.1) there is no advantage in eliminating ρ and ϕ from the differential equation. However, the higher-order metric perturbations turn out to obey inhomogeneous flat-space wave equations of the form

$$\Box\psi = S \tag{2.5}$$

where S is a source term equal to $\mathrm{e}^{im\phi}\rho^{-(n+2)}H(q,r)$. This leads to the following equation for $\chi \equiv \mathrm{e}^{-im\phi}\rho^{n}\psi$:

$$\mathcal{L}_{m,n}\chi(q,r) = H(q,r) \tag{2.6}$$

where $\mathcal{L}_{m,n}$ is an hyperbolic operator in the independent variables q and r, and takes the form [6, 8]

$$\begin{aligned}\mathcal{L}_{m,n} = {} & -(2\sqrt{2} + 32q)\frac{\partial^2}{\partial q \partial r} + 4q^2\frac{\partial^2}{\partial q^2} + 64\frac{\partial^2}{\partial r^2} \\ & + 4(n+1)q\frac{\partial}{\partial q} - 16n\frac{\partial}{\partial r} + n^2 - m^2.\end{aligned} \tag{2.7}$$

The proof of hyperbolicity of $\mathcal{L}_{m,n}$, with the associated normal hyperbolic form, can be found in section 3 of [6], and in [8]. The advantage of studying Eq. (2.6) is twofold: to evaluate the solution at some space-time point one has simply to integrate the product of H and the Green function $G_{m,n}$ of $\mathcal{L}_{m,n}$:

$$\chi(q,r) = \int G_{m,n}(q,r;q_0,r_0)H(q_0,r_0)dq_0dr_0 \tag{2.8}$$

and the resulting numerical calculation of the solution is now feasible [7, 8].

Since we are interested in a direct approach to the evaluation of the Green function $G_{m,n}$ in the (q,r) coordinates, we begin by noticing that, in all derivatives which are not mixed, the operator $\mathcal{L}_{m,n}$ can be made a constant coefficient operator upon setting

$$\alpha \equiv \log(q) \tag{2.9}$$

which implies

$$q\frac{\partial}{\partial q} = \frac{\partial}{\partial \alpha} \tag{2.10}$$

$$q^2\frac{\partial^2}{\partial q^2} = \frac{\partial^2}{\partial \alpha^2} - \frac{\partial}{\partial \alpha} \tag{2.11}$$

and hence

$$\begin{aligned}\mathcal{L}_{m,n} = &-(2\sqrt{2}\mathrm{e}^{-\alpha} + 32)\frac{\partial^2}{\partial\alpha\partial r} + 4\frac{\partial^2}{\partial\alpha^2} + 64\frac{\partial^2}{\partial r^2} \\ &+ 4n\frac{\partial}{\partial\alpha} - 16n\frac{\partial}{\partial r} + n^2 - m^2.\end{aligned} \tag{2.12}$$

This operator is further simplified upon defining the variable

$$R \equiv \frac{r}{4} \tag{2.13}$$

which implies that

$$\begin{aligned}\mathcal{L}_{m,n} = &-\left(\frac{1}{\sqrt{2}}\mathrm{e}^{-\alpha} + 8\right)\frac{\partial^2}{\partial\alpha\partial R} + 4\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial R^2}\right) \\ &+ 4n\left(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial R}\right) + n^2 - m^2.\end{aligned} \tag{2.14}$$

This suggests defining yet new variables

$$X \equiv \alpha + R \tag{2.15}$$

$$Y \equiv \alpha - R \tag{2.16}$$

so that

$$\frac{\partial^2}{\partial\alpha\partial R} = \frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} \tag{2.17}$$

$$\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial R^2} = 2\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right) \tag{2.18}$$

$$\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial R} = 2\frac{\partial}{\partial Y} \tag{2.19}$$

and hence $\mathcal{L}_{m,n}$ reads eventually

$$\begin{aligned} T_{m,n} &= 16\frac{\partial^2}{\partial Y^2} + 8n\frac{\partial}{\partial Y} + n^2 - m^2 \\ &\quad - \frac{1}{\sqrt{2}}\mathrm{e}^{-(X+Y)/2}\left(\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2}\right) \end{aligned} \tag{2.20}$$

with Green function satisfying the equation

$$\begin{aligned} &T_{m,n}G_{m,n}\left(\mathrm{e}^{\frac{X+Y}{2}}, 2(X-Y); \mathrm{e}^{\frac{X_0+Y_0}{2}}, 2(X_0-Y_0)\right) \\ &= \frac{1}{2}\delta\left(\mathrm{e}^{\frac{X+Y}{2}} - \mathrm{e}^{\frac{X_0+Y_0}{2}}\right)\delta((X-Y)-(X_0-Y_0)). \end{aligned} \tag{2.21}$$

The operator $T_{m,n}$ is the sum of an elliptic operator in the Y variable and a two-dimensional wave operator 'weighted' with the exponential $\mathrm{e}^{-(X+Y)/2}$, which is the main source of technical complications in these variables.

3. Reduction to canonical form and the Riemann function

It is therefore more convenient, in our general analysis, to reduce first Eq. (2.6) to canonical form, and then find an integral representation of the solution. Reduction to canonical form means that new coordinates $x = x(q,r)$ and $y = y(q,r)$ are introduced such that the coefficients of $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ vanish. As is shown in [6, 8], this is achieved if

$$\frac{\partial x}{\partial r} = \frac{\partial y}{\partial r} = 1 \tag{3.1}$$

$$\frac{\partial x}{\partial q} = \frac{1 + 8q\sqrt{2} + \sqrt{1+16q\sqrt{2}}}{2\sqrt{2}q^2} \tag{3.2a}$$

$$\frac{\partial y}{\partial q} = \frac{1 + 8q\sqrt{2} - \sqrt{1+16q\sqrt{2}}}{2\sqrt{2}q^2}. \tag{3.3a}$$

The resulting formulae are considerably simplified if one defines

$$t \equiv \sqrt{1+16q\sqrt{2}} = t(x,y). \tag{3.4}$$

The dependence of t on x and y is obtained implicitly by solving the system [6, 8]

$$x = r + \log\left(\frac{t-1}{2}\right) - \frac{8}{(t-1)} - 4 \tag{3.5}$$

$$y = r + \log\left(\frac{t+1}{2}\right) + \frac{8}{(t+1)} - 4. \tag{3.6}$$

This leads to the equation

$$\log\frac{(t-1)}{(t+1)} - \frac{2t}{(t^2-1)} = \frac{(x-y)}{8} \tag{3.7a}$$

which can be cast in the form

$$\frac{(t-1)}{(t+1)} \mathrm{e}^{\frac{2t}{(1-t^2)}} = \mathrm{e}^{\frac{(x-y)}{8}}. \tag{3.7b}$$

This suggests defining

$$w \equiv \frac{(t-1)}{(t+1)} \tag{3.8}$$

so that one first has to solve the transcendental equation

$$w \mathrm{e}^{\frac{(w^2-1)}{2w}} = \mathrm{e}^{\frac{(x-y)}{8}} \tag{3.9}$$

to obtain $w = w(x-y)$, from which one gets

$$t = \frac{(1+w)}{(1-w)} = t(x-y). \tag{3.10}$$

On denoting by $g(w)$ the left-hand side of Eq. (3.9), one finds that, in the plane $(w, g(w))$, the right-hand side of Eq. (3.9) is a line parallel to the w-axis, which intersects $g(w)$ at

no more than one point for each value of $x-y$. For example, when $w=1$, $g(w)$ intersects the line taking the constant value 1, for which $x-y=0$. The function

$$g: w \rightarrow g(w) = w\mathrm{e}^{\frac{(w^2-1)}{2w}}$$

is asymmetric and has the limiting behaviour described by

$$\lim_{w\rightarrow 0^-} g(w) = -\infty \qquad \lim_{w\rightarrow 0^+} g(w) = 0 \tag{3.11}$$

$$\lim_{w\rightarrow -\infty} g(w) = 0 \qquad \lim_{w\rightarrow +\infty} g(w) = \infty. \tag{3.12}$$

Thus, in the lower half-plane, g has an horizontal asymptote given by the w-axis, and a vertical asymptote given by the line $w=0$, while it has no asymptotes in the upper half-plane, since

$$\lim_{w\rightarrow\infty} \frac{g(w)}{w} = \infty$$

in addition to (3.12). The first derivative of g reads

$$g'(w) = \frac{(w+1)^2}{2w}\mathrm{e}^{\frac{(w^2-1)}{2w}}. \tag{3.13}$$

One therefore has $g'(w) > 0$ for all $w > 0$, and $g'(w) < 0$ for all $w \in (-\infty, 0) - \{-1\}$, and g is monotonically decreasing for negative w and monotonically increasing for positive w. The point $w=-1$, at which $g'(w)$ vanishes, is neither a maximum nor a minimum point, because

$$g''(w) = \left(\frac{1}{4w^3} + \frac{1}{2w} + 1 + \frac{w}{4}\right)\mathrm{e}^{\frac{(w^2-1)}{2w}} \tag{3.14}$$

$$g'''(w) = \left(\frac{1}{8w^5} - \frac{3}{4w^4} + \frac{3}{8w^3} + \frac{3}{8w} + \frac{3}{4} + \frac{w}{8}\right)\mathrm{e}^{\frac{(w^2-1)}{2w}}. \tag{3.15}$$

These formulae imply that $g''(-1) = 0$ but $g'''(-1) = -1 \neq 0$, and hence $w=-1$ yields a flex of $g(w)$.

In the (x, y) variables, the operator $\mathcal{L}_{m,n}$ reads therefore

$$\mathcal{L}_{m,n} = f(x, y)\frac{\partial^2}{\partial x \partial y} + g(x, y)\frac{\partial}{\partial x} + h(x, y)\frac{\partial}{\partial y} + n^2 - m^2 \tag{3.16}$$

where, exploiting the formulae

$$\frac{\partial x}{\partial q} = \frac{64\sqrt{2}}{(t-1)^2} \tag{3.2b}$$

$$\frac{\partial y}{\partial q} = \frac{64\sqrt{2}}{(t+1)^2} \tag{3.3b}$$

one finds

$$\begin{aligned} f(x, y) &= -(2\sqrt{2} + 32q)\left(\frac{\partial x}{\partial q} + \frac{\partial y}{\partial q}\right) + 8q^2 \frac{\partial x}{\partial q}\frac{\partial y}{\partial q} + 128 \\ &= 256\left[1 - \frac{2t^2(t^2+1)}{(t-1)^2(t+1)^2}\right] \end{aligned} \tag{3.17}$$

$$g(x, y) = 4(n+1)q\frac{\partial x}{\partial q} - 16n = 16\left[1 + \frac{2(n+1)}{(t-1)}\right] \tag{3.18}$$

$$h(x, y) = 4(n+1)q\frac{\partial y}{\partial q} - 16n = 16\left[1 - \frac{2(n+1)}{(t+1)}\right]. \tag{3.19}$$

The resulting canonical form of Eq. (2.6) is

$$\begin{aligned} L[\chi] &= \left(\frac{\partial^2}{\partial x \partial y} + a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y} + c(x, y)\right)\chi(x, y) \\ &= \widetilde{H}(x, y) \end{aligned} \tag{3.20}$$

where

$$a(x, y) \equiv \frac{g(x, y)}{f(x, y)} = \frac{1}{16}\frac{(1-t)(t+1)^2(2n+1+t)}{(t^4+4t^2-1)} \tag{3.21}$$

$$b(x, y) \equiv \frac{h(x, y)}{f(x, y)} = \frac{1}{16}\frac{(t+1)(t-1)^2(2n+1-t)}{(t^4+4t^2-1)} \tag{3.22}$$

$$c(x, y) \equiv \frac{n^2 - m^2}{f(x, y)} = \frac{(m^2 - n^2)}{256}\frac{(t-1)^2(t+1)^2}{(t^4+4t^2-1)} \tag{3.23}$$

$$\widetilde{H}(x,y) \equiv \frac{H(x,y)}{f(x,y)} = -\frac{H(x,y)}{256}\frac{(t-1)^2(t+1)^2}{(t^4+4t^2-1)}. \tag{3.24}$$

Note that $a(-t) = b(t), b(-t) = a(t), c(-t) = c(t), \widetilde{H}(-t) = \widetilde{H}(t)$.

For an hyperbolic equation in the form (3.20), we can use the Riemann integral representation of the solution. For this purpose, recall from [11] that, on denoting by $L^\dagger$ the adjoint of the operator L in (3.20), which acts according to

$$L^\dagger[\chi] = \chi_{xy} - (a\chi)_x - (b\chi)_y + c\chi \tag{3.25}$$

one has to find a 'function' $R(x, y; \xi, \eta)$ (actually a kernel) subject to the following conditions ((ξ, η) being the coordinates of a point P such that characteristics through it intersect a curve C at points A and B, AP being a segment with constant y, and BP being a segment with constant x):

(i) As a function of x and y, R satisfies the adjoint equation

$$L^\dagger_{(x,y)}[R] = 0 \tag{3.26}$$

(ii) $R_x = bR$ on AP, i.e.

$$R_x(x, y; \xi, \eta) = b(x, \eta)R(x, y; \xi, \eta) \text{ on } y = \eta \tag{3.27}$$

and $R_y = aR$ on BP, i.e.

$$R_y(x, y; \xi, \eta) = a(\xi, y)R(x, y; \xi, \eta) \text{ on } x = \xi \tag{3.28}$$

(iii) R equals 1 at P, i.e.

$$R(\xi, \eta; \xi, \eta) = 1. \tag{3.29}$$

It is then possible to express the solution of Eq. (3.20) in the form

$$\begin{aligned}\chi(P) &= \frac{1}{2}[\chi(A)R(A) + \chi(B)R(B)] \\ &+ \int_{AB}\left(\left[\frac{R}{2}\chi_x + \left(bR - \frac{1}{2}R_x\right)\chi\right]dx\right. \\ &- \left.\left[\frac{R}{2}\chi_y + \left(aR - \frac{1}{2}R_y\right)\chi\right]dy\right) \\ &+ \int\int_\Omega R(x, y; \xi, \eta)\widetilde{H}(x, y)dxdy\end{aligned} \tag{3.30}$$

where Ω is a domain with boundary.

Note that Eqs. (3.27) and (3.28) are ordinary differential equations for the Riemann function $R(x, y; \xi, \eta)$ along the characteristics parallel to the coordinate axes. By virtue of (3.29), their integration yields

$$R(x, \eta; \xi, \eta) = \exp \int_{\xi}^{x} b(\lambda, \eta) d\lambda \tag{3.31}$$

$$R(\xi, y; \xi, \eta) = \exp \int_{\eta}^{y} a(\lambda, \xi) d\lambda \tag{3.32}$$

which are the values of R along the characteristics through P. Equation (3.30) yields instead the solution of Eq. (3.20) for arbitrary initial values given along an arbitrary non-characteristic curve C, by means of a solution R of the adjoint equation (3.26) which depends on x, y and two parameters ξ, η. Unlike χ, the Riemann function R solves a characteristic initial-value problem.

4. Goursat problem for the Riemann function

By fully exploiting the reduction to canonical form of Eq. (2.6) we have considered novel features with respect to the analysis in [6, 8], because the Riemann formula (3.30) contains also the integral along the piece of curve C from A to B, and the term $\frac{1}{2}[\chi(A)R(A) + \chi(B)R(B)]$. This representation of the solution might be more appropriate for the numerical purposes considered in [7], but the task of finding the Riemann function R remains extremely difficult. One can however use approximate methods for solving Eq. (3.26). For this purpose, we first point out that, by virtue of Eq. (3.25), Eq. (3.26) is a canonical hyperbolic equation of the form

$$\left(\frac{\partial^2}{\partial x \partial y} + A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} + C \right) R(x, y; \xi, \eta) = 0 \tag{4.1}$$

where

$$A \equiv -a \tag{4.2}$$

$$B \equiv -b \tag{4.3}$$

$$C \equiv c - a_x - b_y. \tag{4.4}$$

Thus, on defining

$$U \equiv R \tag{4.5}$$

$$V \equiv R_x + BR \tag{4.6}$$

the equation (4.1) for the Riemann function is equivalent to the hyperbolic canonical system [11]

$$U_x = f_1(x,y)U + f_2(x,y)V \tag{4.7}$$

$$V_y = g_1(x,y)U + g_2(x,y)V \tag{4.8}$$

where

$$f_1 \equiv -B = b \tag{4.9}$$

$$f_2 = 1 \tag{4.10}$$

$$g_1 \equiv AB - C + B_y = ab - c + a_x \tag{4.11}$$

$$g_2 \equiv -A = a. \tag{4.12}$$

For the system described by Eqs. (4.7) and (4.8) with boundary data (3.31) and (3.32) an existence and uniqueness theorem holds (see [11] for the Lipschitz conditions on boundary data), and we can therefore exploit the finite differences method to find approximate solutions for the Riemann function $R(x,y;\xi,\eta)$, and eventually $\chi(P)$ with the help of the integral representation (3.30).

5. Concluding remarks

The inverses of hyperbolic operators [12] and the Cauchy problem for hyperbolic equations with polynomial coefficients [13] have always been the object of intensive investigation in the mathematical literature. We have here considered the application of such issues to axisymmetric black hole collisions at the speed of light, relying on the work in [5–8]. We have pointed out that, for the inhomogeneous equations (2.6) occurring in the perturbative analysis, the task of inverting the operator (2.7) can be accomplished with the help of the Riemann integral representation (3.30), after solving Eq. (4.1) for the Riemann function.

One has then to solve a characteristic initial-value problem for a homogeneous hyperbolic equation in canonical form in two independent variables, for which we have developed formulae to be used for the numerical solution with the help of a finite differences scheme. For this purpose one studies the canonical system (cf (4.7) and (4.8))

$$U_x = F(x, y, U, V) \tag{5.1}$$

$$V_y = G(x, y, U, V) \tag{5.2}$$

in the rectangle $\mathcal{R} \equiv \{x, y : x \in [x_0, x_0 + a], y \in [y_0, y_0 + b]\}$ with known values of U on the vertical side AD where $x = x_0$, and known values of V on the horizontal side AB where $y = y_0$. The segments AB and AD are then divided into m and n equal parts, respectively. On setting $\frac{a}{m} \equiv h$ and $\frac{b}{n} \equiv k$, the original differential equations become equations relating values of U and V at three intersection points of the resulting lattice, i.e.

$$\frac{U(P_{r,s+1}) - U(P_{rs})}{h} = F \tag{5.3a}$$

$$\frac{V(P_{r+1,s}) - V(P_{rs})}{k} = G. \tag{5.4a}$$

It is now convenient to set $U_{rs} \equiv U(P_{rs}), V_{rs} \equiv V(P_{rs})$, so that these equations read

$$U_{r,s+1} = U_{rs} + hF(P_{rs}, U_{rs}, V_{rs}) \tag{5.3b}$$

$$V_{r+1,s} = V_{rs} + kG(P_{rs}, U_{rs}, V_{rs}). \tag{5.4b}$$

Thus, if both U and V are known at P_{rs}, one can evaluate U at $P_{r,s+1}$ and V at $P_{r+1,s}$. The evaluation at subsequent intersection points of the lattice goes on along horizontal or vertical segments. In the former case, the resulting algorithm is

$$U_{rs} = U_{r0} + h \sum_{i=1}^{s-1} F(P_{ri}, U_{ri}, V_{ri}) \tag{5.5}$$

$$V_{rs} = V_{r-1,s} + kG(P_{r-1,s}, U_{r-1,s}, V_{r-1,s}) \tag{5.6}$$

while in the latter case one obtains the algorithm expressed by the equations

$$V_{rs} = V_{0s} + \sum_{i=1}^{r-1} G(P_{is}, U_{is}, V_{is}) \tag{5.7}$$

$$U_{rs} = U_{r,s-1} + hF(P_{r,s-1}, U_{r,s-1}, V_{r,s-1}). \tag{5.8}$$

Stability of such solutions is closely linked with the geometry of the associated characteristics, and the criteria to be fulfilled are studied in section 13.2 of [14] (stability depends crucially on whether or not $\frac{h}{k} \leq 1$).

To sum up, one solves numerically Eq. (3.9) for $w = w(x,y) = w(x-y)$, from which one gets $t(x-y)$ with the help of (3.10), which is a fractional linear transformation. This yields a, b, c and $\widetilde{H}$ as functions of (x,y) according to (3.21)–(3.24), and hence A, B and C in the equation for the Riemann function are obtained according to (4.2)–(4.4), where derivatives with respect to x and y are evaluated numerically. Eventually, the system given by (4.7) and (4.8) is solved according to the finite-differences scheme of the present section, with

$$F = f_1 U + f_2 V = f_1 R + f_2(R_x + BR) \tag{5.9}$$

$$G = g_1 U + g_2 V = g_1 R + g_2(R_x + BR). \tag{5.10}$$

Once the Riemann function $R = U$ is obtained with the desired accuracy, numerical evaluation of the integral (3.30) yields $\chi(P)$, and $\chi(q,r)$ is obtained upon using Eqs. (3.5) and (3.6) for the characteristic coordinates. Our steps are conceptually desirable since they rely on well established techniques for the solution of hyperbolic equations in two independent variables [11, 14], and provide a viable alternative to the numerical analysis performed in [7], because all functions should be evaluated numerically. Our method is not obviously more powerful than the one used in [5–8], but is well suited for a systematic and lengthy numerical analysis, while its analytic side provides an interesting alternative for the evaluation of Green functions both in black hole physics and in other problems where hyperbolic operators with variable coefficients might occur. This task remains very important because a strong production of gravitational radiation is mainly expected in the

extreme events studied in [5–8] and which motivated our paper. Any viable way of looking at mathematical and numerical aspects of the problem is therefore of physical interest for research planned in the years to come [15].

Acknowledgments

The work of G Esposito has been partially supported by the National project *SINTESI 2000*.

References

[1] DeWitt B S 1965 *Dynamical Theory of Groups and Fields* (New York: Gordon and Breach)

[2] DeWitt B S 1984 in *Relativity, Groups and Topology II* eds B S DeWitt and R Stora (Amsterdam: North–Holland)

[3] Esposito G 2001 *Quantum Gravity in Four Dimensions* (New York: Nova Science)

[4] Cartier P and DeWitt–Morette C 2000 *J. Math. Phys.* **41** 4154

[5] D'Eath P D and Payne P N 1992 *Phys. Rev.* D **46** 658

[6] D'Eath P D and Payne P N 1992 *Phys. Rev.* D **46** 675

[7] D'Eath P D and Payne P N 1992 *Phys. Rev.* D **46** 694

[8] D'Eath P D 1996 *Black Holes: Gravitational Interactions* (Oxford: Clarendon Press)

[9] Esposito G and Stornaiolo C 2000 *Class. Quantum Grav.* **17** 1989

[10] Esposito G and Stornaiolo C 2000 *Found. Phys. Lett.* **13** 279

[11] Courant R and Hilbert D 1961 *Methods of Mathematical Physics. II. Partial Differential Equations* (New York: Interscience)

[12] Leray J 1953 *Hyperbolic Differential Equations* (Princeton University Press)

[13] Leray J 1956 *C. R. Acad. Sci. Paris* **242** 953

[14] Garabedian P R 1964 *Partial Differential Equations* (New York: Chelsea)

[15] Allen B and Ottewill A (2000) *Gen. Rel. Grav.* **32** 385